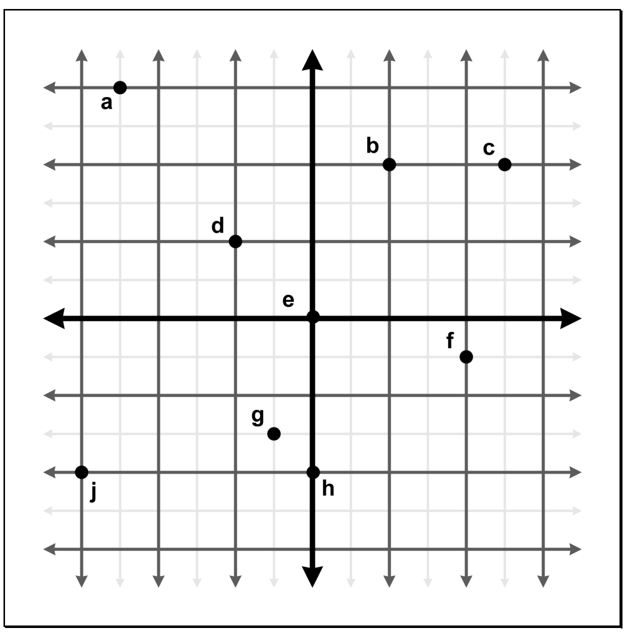
Answers

Chapter 2

1) *Give the coordinates of the following points:*



a	(-2.5, 3)
b	(1, 2)
c	(2.5, 2)
d	(-1, 1)
e	(0, 0)
f	(2, -0.5)
g	(-0.5, -1.5)
h	(0, -2)
j	(-3, -2)

2) *List the 48 different possible ways that the 3D axes may be assigned to the directions "north," "east" and "up." Identify which of these combinations are left-handed, and which are right-handed.*

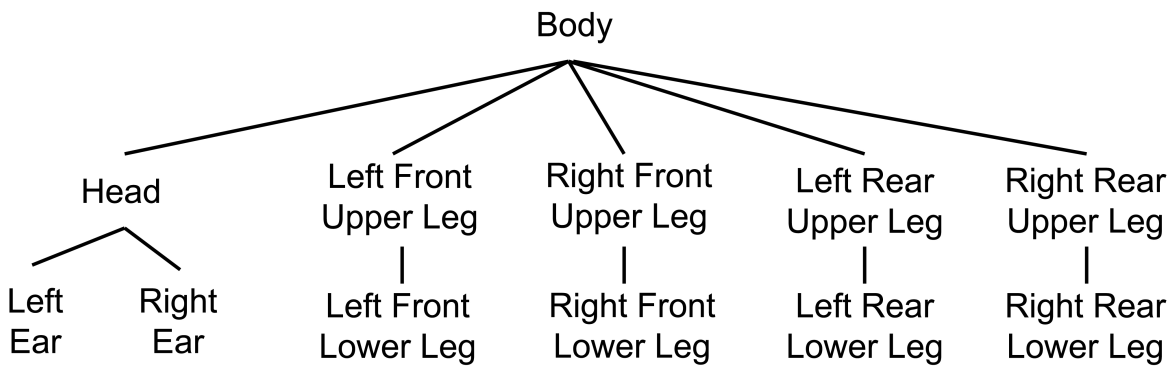
North	East	Up	Hand	North	East	Up	Hand
+x	+y	+z	Left	+x	+z	+y	Right
+x	+y	–z	Right	+x	+z	–y	Left
+x	–y	+z	Right	+x	–z	+y	Left
+x	–y	–z	Left	+x	–z	–y	Right
–x	+y	+z	Right	–x	+z	+y	Left
–x	+y	–z	Left	–x	+z	–y	Right
–x	–y	+z	Left	–x	–z	+y	Right
–x	–y	–z	Right	–x	–z	–y	Left
+y	+x	+z	Right	+y	+z	+x	Left
+y	+x	–z	Left	+y	+z	–x	Right
+y	–x	+z	Left	+y	–z	+x	Right
+y	–x	–z	Right	+y	–z	–x	Left
–y	+x	+z	Left	–y	+z	+x	Right
–y	+x	–z	Right	–y	+z	–x	Left
–y	–x	+z	Right	–y	–z	+x	Left
–y	–x	–z	Left	–y	–z	–x	Right
+z	+x	+y	Left	+z	+y	+x	Right
+z	+x	–y	Right	+z	+y	–x	Left
+z	–x	+y	Right	+z	–y	+x	Left
+z	–x	–y	Left	+z	–y	–x	Right
–z	+x	+y	Right	–z	+y	+x	Left
–z	+x	–y	Left	–z	+y	–x	Right
–z	–x	+y	Left	–z	–y	+x	Right
–z	–x	–y	Right	–z	–y	–x	Left

3) *In a popular modeling program 3D Studio Max, the default orientation of the axes is for +x to point right, +y to point forward, and +z to point up. Is this a left- or right-handed coordinate space?*

Right-handed.

Chapter 3

1) *Draw a nested space hierarchy tree for the sheep described in Section 3.3, assuming that its head, ears, upper legs, lower legs, and body move indemendently.*



2) *Suppose our object axes are transformed to world axes by rotating them counterclockwise around the* y*-axis by 42° and then translating six units along the* z*-axis and 12 units along the* x*-axis. Describe this transformation from the perspective of the object.*

Imagine a point on the object, in object space. As the axes are rotating counterclockwise, the point is actually rotating *counterclockwise* relative to the axes. Then, as the axes translate by [12, 0, 6], the point translates [-12, 0, -6] relative to the axes.

3) *Which coordinate space is the most appropriate in which to ask the following questions?*

a) *Is my computer in front of or behind me?* Object space. If we know the position of the computer within our object space, this question is a trivial matter of checking for a positive z value. (Assuming the conventions from Section 2.3.4)

b) *Is the book east or west of me?* Inertial space is the easiest space to make this test. Again, assuming the conventions from Section 2.3.4, the book is east of us if the x-coordinate of the book's position in our inertial space is positive, and west if this value is negative. Alternatively, we could answer the question in world space, by comparing the x-coordinate of the book in world space, with our own world space x-coordinate.

c) *How do I get from one room to the other?* Pathfinding-type querries are usually made in world space.

d) *Can I see my computer?* The "camera space" for our viewpoint is the most natural coordinate space to use for this question.

Chapter 4

1) *Let:*

$$\mathbf{a} = \begin{bmatrix} -3 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 16 \\ -1 \\ 4 \\ 6 \end{bmatrix}$$

a) *Identify* **a**, **b**, *and* **c**, *as row or column vectors, and give the dimension of each vector.*

a is a 2D row vector. **b** is a 3D column vector. **c** is a 4D column vector.

b) *Compute* $\mathbf{b}_y+\mathbf{c}_w+\mathbf{a}_x+\mathbf{b}_z$.

$$\begin{aligned} \mathbf{b}_y &= 0 \\ \mathbf{c}_w &= 6 \\ \mathbf{a}_x &= -3 \\ \mathbf{b}_z &= 5 \\ \mathbf{b}_y + \mathbf{c}_w + \mathbf{a}_x + \mathbf{b}_z &= 0 + 6 + (-3) + 5 \\ &= 8 \end{aligned}$$

2) *Identity the quantities in each of the following sentences as scalar or vector. For vector quantities, give the magnitude and direction. (Note: some directions may be implicit.)*

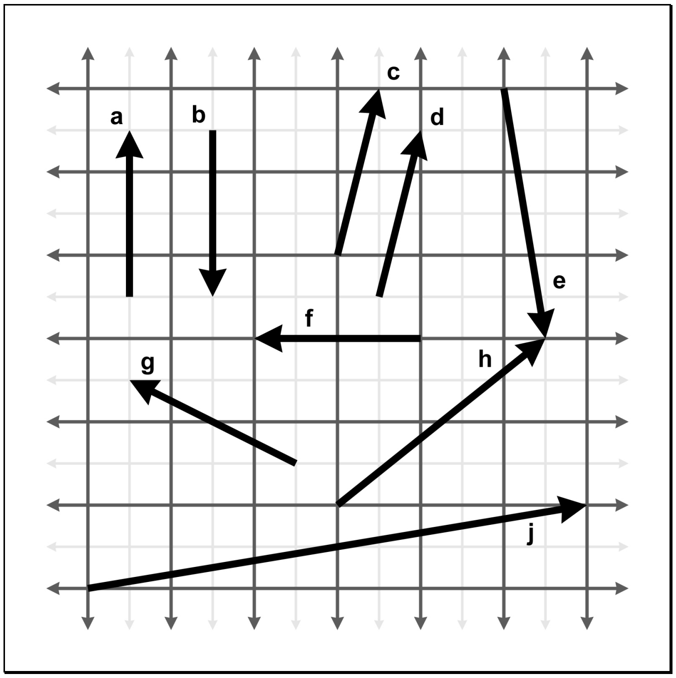
a) *How much do you weigh?* Weight is a calar quantity.

b) *Do you have any idea how fast you were going?* Speed is a scalar quantity.

c) *It's two blocks north of here.* "Two blocks north" is a vector quantity, since it specified a magnitude ("two blocks") and a direction ("north").

d) *We're cruising from Los Angeles to New York at 600mph, at an altitude of 33,000ft.* Speed (600mph) is a scalar quantity. However, since we know we are traveling from Los Angeles to New York, we could assume an eastward direction, which would provide a direction, making it a velocity, which is a vector quantity. Altitude (33,000ft) is a scalar quantity.

3) *Give the values of the following vectors:*



a	[0, 2]
b	[0, -2]
c	[0.5, 2]
d	[0.5, 2]
e	[0.5, -3]
f	[-2, 0]
g	[-2, 1]
h	[2.5, 2]
j	[6, 1]

4) *Identify the following statements as true or false. If the statement is false, explain why.*

a) *The size of a vector in a diagram doesn't matter; we just need to draw it in the right place.* **False**. Size matters; so does direction. A vector does not express a "position," and so we can draw in on a diagram anywhere that is convenient. See Section 4.2.2.

b) *The displacement expressed by a vector can be visualized as a sequence of axially aligned displacements*. **True**. See Figure 4.5 on page 40.

c) *These axially aligned displacements from the previous question must occur in order.* **False**. They can occur in any order, due to commutative nature of vector addition. See page 40.

d) *The vector [x, y] gives the displacement from the point (x, y) to the origin.* **False**. It gives the opposite displacement – from the origin to the point.

Chapter 5

1) *Evaluate the following vector expressions:*

a) $-\begin{bmatrix} 3 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -7 \end{bmatrix}$

b)

$$\begin{aligned} \left\| \begin{bmatrix} -12 & 5 \end{bmatrix} \right\| &= \sqrt{(-12)^2 + 5^2} \\ &= \sqrt{144 + 25} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

c)

$$\begin{aligned} \left\| \begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix} \right\| &= \sqrt{8^2 + (-3)^2 + (\frac{1}{2})^2} \\ &= \sqrt{64 + 9 + \frac{1}{4}} \\ &= \sqrt{64 + 9 + \frac{1}{4}} \\ &= \sqrt{73\frac{1}{4}} \\ &\approx 8.5586 \end{aligned}$$

d) $3\begin{bmatrix} 4 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 4 & 3 \cdot -7 & 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 12 & -21 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 4 & 5 \end{bmatrix}/2 = \begin{bmatrix} 4/2 & 5/2 \end{bmatrix} = \begin{bmatrix} 2 & 5/2 \end{bmatrix}$

2) *Normalize the following vectors:*

a)

$$\begin{aligned} \begin{bmatrix} 12 & 5 \end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix} 12 & 5 \end{bmatrix}}{\left\| \begin{bmatrix} 12 & 5 \end{bmatrix} \right\|} \\ &= \frac{\begin{bmatrix} 12 & 5 \end{bmatrix}}{\sqrt{12^2 + 5^2}} \\ &= \frac{\begin{bmatrix} 12 & 5 \end{bmatrix}}{13} \\ &= \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \end{bmatrix} \end{aligned}$$

b)

$$\begin{aligned} \begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix}}{\left\| \begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix} \right\|} \\ &= \frac{\begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix}}{\sqrt{8^2 + (-3)^2 + (1/2)^2}} \\ &= \frac{\begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix}}{\sqrt{64 + 9 + (1/4)}} \\ &= \frac{\begin{bmatrix} 8 & -3 & 1/2 \end{bmatrix}}{\sqrt{293/4}} \\ &\approx \frac{\begin{bmatrix} 8 & -3 & .5 \end{bmatrix}}{8.5586} \\ &\approx \begin{bmatrix} .9347 & -.3505 & .05842 \end{bmatrix} \end{aligned}$$

3) *Evaluate the following vector expressions:*

a)

$$\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} - \begin{bmatrix} 8 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 - 8 \\ 10 - (-7) \\ 7 - 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 17 \\ 3 \end{bmatrix}$$

b)

$$3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 10 \\ -6 \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \\ 3c \end{bmatrix} - \begin{bmatrix} 8 \\ 40 \\ -24 \end{bmatrix} = \begin{bmatrix} 3a - 8 \\ 3b - 40 \\ 3c + 24 \end{bmatrix}$$

4) *Compute the distance between the following pairs of points:*

a)

$$
\begin{aligned}
\text{distance}\left(\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ 4 \end{bmatrix}\right) &= \sqrt{(3-8)^2 + (10-(-7))^2(7-4)^2} \\
&= \sqrt{(-5)^2 + 17^2(-3)^2} \\
&= \sqrt{25 + 289 + 9} \\
&= \sqrt{323} \\
&\approx 17.9722
\end{aligned}
$$

b)

$$
\begin{aligned}
\text{distance}\left(\begin{bmatrix} 10 \\ 6 \end{bmatrix}, \begin{bmatrix} -14 \\ 30 \end{bmatrix}\right) &= \sqrt{(10-(-14))^2 + (6-30)^2} \\
&= \sqrt{24^2 + (-24)^2} \\
&= \sqrt{576 + 576} \\
&= \sqrt{1152} \\
&= 24\sqrt{2} \\
&\approx 33.9411
\end{aligned}
$$

5) *Evaluate the following vector expressions:*

a) $$\begin{bmatrix} 2 \\ 6 \end{bmatrix} \cdot -38 = \begin{bmatrix} (2)(-38) \\ (6)(-38) \end{bmatrix} = \begin{bmatrix} -76 \\ -228 \end{bmatrix}$$

Note: Although the above problem is valid, the notation isn't the same as the notation used in the book. That's because it contained a typo. The problem should have read:

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 8 \end{bmatrix} = (2)(-3) + (6)(8) = -6 + 48 = 42$$

b)

$$
\begin{aligned}
3\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 8 \\ -2 \\ 3/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \\ 7 \end{bmatrix} \right) &= \begin{bmatrix} (3)(-2) \\ (3)(0) \\ (3)(4) \end{bmatrix} \cdot \begin{bmatrix} 8+0 \\ -2+9 \\ 3/2+7 \end{bmatrix} \\
&= \begin{bmatrix} -6 \\ 0 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 7 \\ 17/2 \end{bmatrix} \\
&= (-6)(8) + (0)(7) + (12)(17/2) \\
&= -48 + 0 + 102 \\
&= 54
\end{aligned}
$$

6) *Compute the angle between the vectors [1, 2] and [-6, 3].*

From Section 5.10.2, we solve for the angle using the dot product:

$$
\begin{aligned}
\theta &= \text{acos}\left(\frac{\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -6 & 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 & 2 \end{bmatrix} \right\| \left\| \begin{bmatrix} -6 & 3 \end{bmatrix} \right\|} \right) \\
&= \text{acos}\left(\frac{(1)(-6) + (2)(3)}{\sqrt{1^2 + 2^2}\sqrt{(-6)^2 + 3^2}} \right) \\
&= \text{acos}\left(\frac{-6+6}{\sqrt{5}\sqrt{45}} \right) \\
&= \text{acos}\ 0 \\
&= 90^{\circ}
\end{aligned}
$$

7) *Given the two vectors*

$$\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}, \mathbf{n} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

*Separate **v** into components that are perpendicular and parallel to **n**. (**n** is a unit vector.)*

See Section 5.10.3.

$$\begin{aligned}
\mathbf{v}_{\parallel} &= \mathbf{n}\frac{\mathbf{v}\cdot\mathbf{n}}{\|\mathbf{n}\|^2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \frac{\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}}{1^2} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \left(\frac{4\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + (-1)(0) \right) \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \frac{7\sqrt{2}}{2} = \begin{bmatrix} (\frac{\sqrt{2}}{2})(\frac{7\sqrt{2}}{2}) \\ (\frac{\sqrt{2}}{2})(\frac{7\sqrt{2}}{2}) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 7/2 \\ 7/2 \\ 0 \end{bmatrix} \\
\mathbf{v}_{\perp} &= \mathbf{v} - \mathbf{v}_{\parallel} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 7/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix}
\end{aligned}$$

8) *Compute the value of*

$$\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} \times \begin{bmatrix} 8 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} (10)(4) - (7)(-7) \\ (7)(8) - (3)(4) \\ (3)(-7) - (10)(8) \end{bmatrix} = \begin{bmatrix} 40 - (-49) \\ (56) - (12) \\ (-21) - (80) \end{bmatrix} = \begin{bmatrix} 89 \\ 44 \\ -101 \end{bmatrix}$$

9) *A man is boarding a plane. The airline has a rule where no carry-on item may be more than 2ft long, 2ft wide, or 2ft tall. He has a very valuable sword that is three feet long. He is able to carry the sword on board with him. How is he able to do this? What is the longest possible item that he could carry on?*

The man is able to board the plane by placing his sword diagonally in a cube-shaped box that is 2ft long, 2ft tall, and 2ft wide. The length of the longest item he could carry is:

$$\left\| \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \right\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3} \approx 3.4641$$

which is about 41.5 inches. (Of course, nowadays, he would be arrested and would not be allowed to board the plane at all! This question was written before the recent increase airport security.)

10) *Verify Figure 5.7 on page 56 mathematically.*

$$\begin{aligned} &\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e} + \mathbf{f} \\ &= \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -1 + 1 + 3 + (-1) + (-6) + (-1) \\ 3 + 3 + (-2) + (-2) + 4 + (-3) \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ 3 \end{bmatrix} \end{aligned}$$

11) *Is the coordinate system used in Figure 5.13 on page 63 a left-handed or right-handed coordinate system?*

Left-handed.

12) *Assume that Texas is flat. A minute of latitude is approximately 1.15 miles in length. At the authors' latitude (see section 3.2.1), a minute of longitude is approximately 0.97 miles in length. There are 60 minutes in one degree of latitude or longitude. How far apart are the authors?*

First, we need to convert degrees and minutes to miles. The latitudinal and longitudinal distances are:

$$\begin{array}{llcccl} \text{Latitude} & 33^\circ 11' - 33^\circ 01' & = & 10' & \approx & 11.5\text{miles} \\ \text{Longitude} & 97^\circ 07' - 96^\circ 59' & = & 8' & \approx & 7.76\text{miles} \end{array}$$

Now, we apply the 2D distance formula (Equation 5.12):

Chapter 7

1) *Given the following matrices:*

$$\mathbf{A} = \begin{bmatrix} 13 & 4 & -8 \\ 12 & 0 & 6 \\ -3 & -1 & 5 \\ 10 & -2 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 15 & 8 \\ -7 & 3 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} a & g \\ b & h \\ c & i \\ d & j \\ f & k \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

a) *For each matrix **A** through **F** above, give the dimensions of the matrix and identify the matrix as square and/or diagonal.*

Matrix	Dimensions	Square	Diagonal
A	4×3	No	No
B	2×2	Yes	Yes
C	2×2	Yes	No
D	1×3	No	No
E	5×2	No	No
F	4×1	No	No

b) *Determine if the following matrix multiplications are allowed, and if so, give the dimensions of the resulting matrix.*

Product	Dimensions
DA	Undefined
AD	Undefined
BC	2×2
AF	Undefined
E^TB	Undefined
DFA	Undefined

c) *Compute the following transpositions:*

$$\mathbf{A}^T = \begin{bmatrix} 13 & 4 & -8 \\ 12 & 0 & 6 \\ -3 & -1 & 5 \\ 10 & -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 13 & 12 & -3 & 10 \\ 4 & 0 & -1 & -2 \\ -8 & 6 & 5 & 5 \end{bmatrix}$$

$$\mathbf{E}^T = \begin{bmatrix} a & g \\ b & h \\ c & i \\ d & j \\ f & k \end{bmatrix}^T = \begin{bmatrix} a & b & c & d & f \\ g & h & i & j & k \end{bmatrix}$$

$$\mathbf{B}^T = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}^T = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$$

2) *Compute the following products:*

a)

$$\begin{bmatrix} 1 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 7 \\ 4 & 1/3 \end{bmatrix} = \begin{bmatrix} (1)(-3)+(-2)(4) & (1)(7)+(-2)(1/3) \\ (5)(-3)+(0)(4) & (5)(7)+(0)(1/3) \end{bmatrix}$$

$$= \begin{bmatrix} -3+(-8) & 7+(-2/3) \\ -15+0 & 35+0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 19/3 \\ -15 & 35 \end{bmatrix}$$

b)

$$\begin{bmatrix} 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 3 \\ 5 & 7 & -6 \\ 1 & -4 & 2 \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} (3)(-2)+(-1)(5)+(4)(1) & (3)(0)+(-1)(7)+(4)(-4) & (3)(3)+(-1)(-6)+(4)(2) \end{bmatrix} \\ &= \begin{bmatrix} (-6)+(-5)+4 & 0+(-7)+(-16) & 9+6+8 \end{bmatrix} \\ &= \begin{bmatrix} -7 & -23 & 23 \end{bmatrix} \end{aligned}$$

3) *Manipulate the following matrix product to remove the parenthesis:*

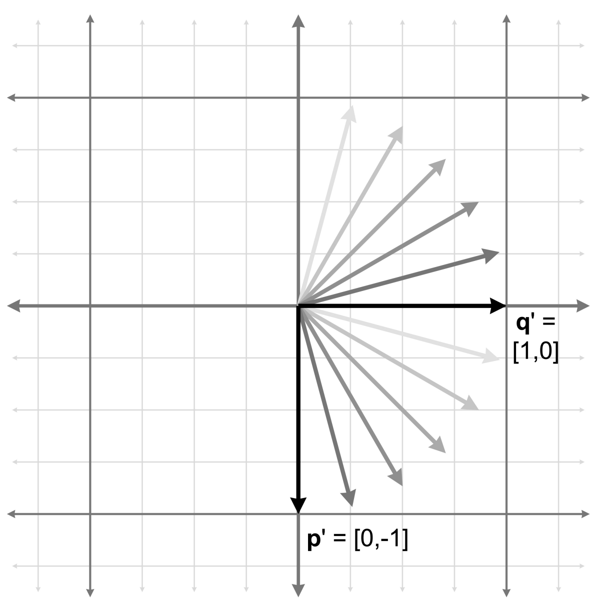
$$\left((\mathbf{AB})^T (\mathbf{CDE})^T\right)^T$$

$$\begin{aligned} &= \left((\mathbf{CDE})^T\right)^T \left((\mathbf{AB})^T\right)^T \\ &= (\mathbf{CDE})(\mathbf{AB}) \\ &= \mathbf{CDEAB} \end{aligned}$$

4) *What type of transformation is represented by the following 2D matrix:*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Extracting the basis vectors [0,-1] and [1,0] and drawing them on a coordinate grid, we see that the transformation matrix performs a clockwise rotation about the origin by 90 degrees.



Chapter 8

1) *Construct a matrix to rotate -22° about the* x-*axis.*

Using Equation 8.2 on page 108:

$$\mathbf{R}_x(-22^o) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos -22^o & \sin -22^o \\ 0 & -\sin -22^o & \cos -22^o \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & .927 & -.375 \\ 0 & .375 & .927 \end{bmatrix}$$

2) *Construct a matrix to rotate 30° about the* y-*axis.*

Using Equation 8.3 on page 108:

$$\mathbf{R}_y(30^o) = \begin{bmatrix} \cos 30^o & 0 & -\sin 30^o \\ 0 & 1 & 0 \\ \sin 30^o & 0 & \cos 30^o \end{bmatrix} \approx \begin{bmatrix} .866 & 0 & -.500 \\ 0 & 1 & 0 \\ .500 & 0 & .866 \end{bmatrix}$$

3) *Construct a matrix that transforms a vector from inertial space to object space. From the "identity orientation," the object rotated 30° around its* y-*axis and then -22° about its* x-*axis.*

This is a trick question, intended to make you think about exactly what happens when a vector is rotated from inertial to object space. The most tempting error is to just take the two matrices from the previous sections and concatenate them in order. But let's think about exactly what is happening.

Remember that when we transform a vector from one coordinate space to another, the vector doesn't actually move, we are just expressing it a different coordinate space. So let's imagine the coordinate space itself rotating with the object from inertial to object space. First, the object (and its coordinate space) rotated 30° around the y-axis. Now, as the coordinate space rotates positive 30°, a vector would rotate *negative* 30° (relative to the coordinate space – remember, the vector is actually stationary). Likewise, when the object (and the coordinate space) rotates -22° about the x-axis, a vector will rotate positive 22° (relative to the coordinate space). Now the coordinate space is in line with object space, and our vector is expressed in object space.

$$\begin{aligned}
\mathbf{R}_y(-30^o) &= \begin{bmatrix} \cos -30^o & 0 & -\sin -30^o \\ 0 & 1 & 0 \\ \sin -30^o & 0 & \cos -30^o \end{bmatrix} \approx \begin{bmatrix} .866 & 0 & .500 \\ 0 & 1 & 0 \\ -.500 & 0 & .866 \end{bmatrix} \\
\mathbf{R}_x(22^o) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 22^o & \sin 22^o \\ 0 & -\sin 22^o & \cos 22^o \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & .927 & .375 \\ 0 & -.375 & .927 \end{bmatrix} \\
\mathbf{M}_{inertial \to object} &= \mathbf{R}_y(-30^o)\mathbf{R}_x(22^o) \\
&\approx \begin{bmatrix} .866 & 0 & .500 \\ 0 & 1 & 0 \\ -.500 & 0 & .866 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & .927 & .375 \\ 0 & -.375 & .927 \end{bmatrix} \\
&\approx \begin{bmatrix} .866 & -.188 & .464 \\ 0 & .927 & .375 \\ -.500 & -.325 & .803 \end{bmatrix}
\end{aligned}$$

4) *Express the object's* z-*axis using inertial coordinates.*

The object's z-axis in object space is trivially [0, 0, 1]. Our task is to transform this vector into inertial space. To do this, we will construct the object-to-inertial matrix. This is the opposite of the inertial-to-object matrix. Recall from the previous exercise that the inertial-to-object matrix first rotates about the y-axis by -30˚ and then about the x-axis by 22˚. The object-to-inertial matrix will do the opposite: we will first rotate about the x-axis by -22˚, and then about the y-axis by 30˚. We can get the values for the matrix from the first two exercises.

$$\begin{aligned}
\mathbf{R}_x(-22^o) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos -22^o & \sin -22^o \\ 0 & -\sin -22^o & \cos -22^o \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & .927 & -.375 \\ 0 & .375 & .927 \end{bmatrix} \\
\mathbf{R}_y(30^o) &= \begin{bmatrix} \cos 30^o & 0 & -\sin 30^o \\ 0 & 1 & 0 \\ \sin 30^o & 0 & \cos 30^o \end{bmatrix} \approx \begin{bmatrix} .866 & 0 & -.500 \\ 0 & 1 & 0 \\ .500 & 0 & .866 \end{bmatrix} \\
\mathbf{M}_{object \to inertial} &= \mathbf{R}_x(-22^o)\mathbf{R}_y(30^o) \\
&\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & .927 & -.375 \\ 0 & .375 & .927 \end{bmatrix} \begin{bmatrix} .866 & 0 & -.500 \\ 0 & 1 & 0 \\ .500 & 0 & .866 \end{bmatrix} \\
&\approx \begin{bmatrix} .866 & 0 & -.500 \\ -.188 & .927 & -.325 \\ .464 & .375 & .803 \end{bmatrix}
\end{aligned}$$

Notice that the object-to-inertial matrix is the transpose of the inertial-to-object matrix that we computed in the previous exercise. Also, notice that the rotation matrices that rotate about a single axes are the transpose of the corresponding matrices that rotate about the same axis by the opposite rotation angle. Any rotation matrix is "orthogonal," which means that the inverse matrix (the matrix which does the "opposite" rotation) is obtained simply by transposing the matrix. Matrix inversion and orthogonal matrices are discussed in detail Sections 9.2 and 9.3.

Now that we have the object-to-inertial matrix, we can compute the unit vector corresponding to the *z*-axis, by transforming the vector [0, 0, 1] from object to inertial space:

$$\begin{aligned}
\mathbf{z}_{inertial} &= \mathbf{z}_{object}\mathbf{M}_{object\rightarrow inertial} \\
&\approx \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .866 & 0 & -.500 \\ -.188 & .927 & -.325 \\ .464 & .375 & .803 \end{bmatrix} \\
&\approx \begin{bmatrix} -.500 & -.325 & .803 \end{bmatrix}
\end{aligned}$$

5) *Construct a matrix to rotate 164° about the* z-*axis.*

Using Equation 8.4 on page 109:

$$\mathbf{R}_z(164^\circ) = \begin{bmatrix} \cos 164^\circ & \sin 164^\circ & 0 \\ -\sin 164^\circ & \cos 164^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} -.961 & .276 & 0 \\ -.276 & -.961 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6) *Construct a matrix to rotate -5° about the axis [99, -99, 99].*

We will use Equation 8.5 on page 111. However, this requires that our axis of rotation, **n**, be a unit vector. So we first normalize the vector [99, -99, 99] to calculate **n**.

$$\begin{aligned}
\mathbf{n} &= \begin{bmatrix} 99 & -99 & 99 \end{bmatrix} / \left\| \begin{bmatrix} 99 & -99 & 99 \end{bmatrix} \right\| \\
&= \begin{bmatrix} 99 & -99 & 99 \end{bmatrix} / \sqrt{99^2 + (-99)^2 + 99^2} \\
&= \begin{bmatrix} 99 & -99 & 99 \end{bmatrix} / 171.47 \\
&= \begin{bmatrix} .577 & -.577 & .577 \end{bmatrix}
\end{aligned}$$

Now we can apply Equation 8.5 directly:

$$
\begin{aligned}
\mathbf{n} &= \begin{bmatrix} .577 & -.577 & .577 \end{bmatrix} \\
\theta &= -5^{\circ} \\
\mathbf{R}(\mathbf{n},\theta) &= \begin{bmatrix} {\mathbf{n}_x}^2\left(1-\cos\theta\right)+\cos\theta & \mathbf{n}_x\mathbf{n}_y\left(1-\cos\theta\right)+\mathbf{n}_z\sin\theta & \mathbf{n}_x\mathbf{n}_z\left(1-\cos\theta\right)-\mathbf{n}_y\sin\theta \\ \mathbf{n}_x\mathbf{n}_y\left(1-\cos\theta\right)-\mathbf{n}_z\sin\theta & {\mathbf{n}_y}^2\left(1-\cos\theta\right)+\cos\theta & \mathbf{n}_y\mathbf{n}_z\left(1-\cos\theta\right)+\mathbf{n}_x\sin\theta \\ \mathbf{n}_x\mathbf{n}_z\left(1-\cos\theta\right)+\mathbf{n}_y\sin\theta & \mathbf{n}_y\mathbf{n}_z\left(1-\cos\theta\right)-\mathbf{n}_x\sin\theta & {\mathbf{n}_z}^2\left(1-\cos\theta\right)+\cos\theta \end{bmatrix} \\
&\approx \begin{bmatrix} (.577)^2\left(1-\cos -5^{\circ}\right)+\cos -5^{\circ} & (.577)(-.577)\left(1-\cos -5^{\circ}\right)+(.577)\sin -5^{\circ} & (.577)(.577)\left(1-\cos -5^{\circ}\right)-(-.577)\sin -5^{\circ} \\ (.577)(-.577)\left(1-\cos -5^{\circ}\right)-(.577)\sin -5^{\circ} & (-.577)^2\left(1-\cos -5^{\circ}\right)+\cos -5^{\circ} & (-.577)(.577)\left(1-\cos -5^{\circ}\right)+(.577)\sin -5^{\circ} \\ (.577)(.577)\left(1-\cos -5^{\circ}\right)+(-.577)\sin -5^{\circ} & (-.577)(.577)\left(1-\cos -5^{\circ}\right)-(.577)\sin -5^{\circ} & (.577)^2\left(1-\cos -5^{\circ}\right)+\cos -5^{\circ} \end{bmatrix} \\
&\approx \begin{bmatrix} .333(.00381)+.996 & -.333(.00381)+(.577)(-.00872) & .333(.00381)-(-.577)(-.00872) \\ -.333(.00381)-(.577)(-.00872) & .333(.00381)+.996 & -.333(.00381)+(.577)(-.00872) \\ .333(.00381)+(-.577)(-.00872) & -.333(.00381)-(.577)(-.00872) & .333(.00381)+.996 \end{bmatrix} \\
&\approx \begin{bmatrix} .997 & -.00630 & -.00376 \\ .00376 & .997 & -.00630 \\ .00630 & .00376 & .997 \end{bmatrix}
\end{aligned}
$$

7) *Construct a matrix that doubles the height, width, and length of an object.*

Using Equation 8.7 from page 113:

$$
\mathbf{S}(2,2,2) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}
$$

8) *Construct a matrix to scale by a factor of 5 about the plane through the origin perpendicular to the vector [99, -99, 99].*

We will use Equation 8.9 from page 115, which requires that our perpendicular vector, **n**, be normalized. We computed this unit vector in exercise 6.

$$
\begin{aligned}
\mathbf{n} &= \begin{bmatrix} .577 & -.577 & .577 \end{bmatrix} \\
k &= 5 \\
\mathbf{S}(\mathbf{n},k) &= \begin{bmatrix} 1+(k-1)\,{\mathbf{n}_x}^2 & (k-1)\,\mathbf{n}_x\mathbf{n}_y & (k-1)\,\mathbf{n}_x\mathbf{n}_z \\ (k-1)\,\mathbf{n}_x\mathbf{n}_y & 1+(k-1)\,{\mathbf{n}_y}^2 & (k-1)\,\mathbf{n}_y\mathbf{n}_z \\ (k-1)\,\mathbf{n}_x\mathbf{n}_z & (k-1)\,\mathbf{n}_y\mathbf{n}_z & 1+(k-1)\,{\mathbf{n}_z}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1+(5-1)\,(.577)^2 & (5-1)\,(.577)(-.577) & (5-1)\,(.577)(.577) \\ (5-1)\,(.577)(-.577) & 1+(5-1)\,(-.577)^2 & (5-1)\,(-.577)(.577) \\ (5-1)\,(.577)(.577) & (5-1)\,(-.577)(.577) & 1+(5-1)\,(.577)^2 \end{bmatrix} \\
&= \begin{bmatrix} 2.33 & -1.33 & 1.33 \\ -1.33 & 2.33 & -1.33 \\ 1.33 & -1.33 & 2.33 \end{bmatrix}
\end{aligned}
$$

9) *Construct a matrix to orthographically project onto the plane through the origin perpendicular to the vector [99, -99, 99].*

We apply Equation 8.16 from page 117.

$$
\begin{aligned}
\mathbf{n} &= \begin{bmatrix} .577 & -.577 & .577 \end{bmatrix} \\
\mathbf{P}(\mathbf{n}) &= \begin{bmatrix} 1-{\mathbf{n}_x}^2 & -\mathbf{n}_x\mathbf{n}_y & -\mathbf{n}_x\mathbf{n}_z \\ -\mathbf{n}_x\mathbf{n}_y & 1-{\mathbf{n}_y}^2 & -\mathbf{n}_y\mathbf{n}_z \\ -\mathbf{n}_x\mathbf{n}_z & -\mathbf{n}_y\mathbf{n}_z & 1-{\mathbf{n}_z}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1-(.577)^2 & -(.577)(-.577) & -(.577)(.577) \\ -(.577)(-.577) & 1-(-.577)^2 & -(-.577)(.577) \\ -(.577)(.577) & -(-.577)(.577) & 1-(.577)^2 \end{bmatrix} \\
&= \begin{bmatrix} .667 & .333 & -.333 \\ .333 & .667 & .333 \\ -.333 & .333 & .667 \end{bmatrix}
\end{aligned}
$$

10) *Construct a matrix to reflect orthographically about the plane through the origin perpendicular to the vector [99, -99, 99].*

We apply Equation 8.18 from page 118.

$$
\begin{aligned}
\mathbf{n} &= \begin{bmatrix} .577 & -.577 & .577 \end{bmatrix} \\
\mathbf{R}(\mathbf{n}) &= \begin{bmatrix} 1-2{\mathbf{n}_x}^2 & -2\mathbf{n}_x\mathbf{n}_y & -2\mathbf{n}_x\mathbf{n}_z \\ -2\mathbf{n}_x\mathbf{n}_y & 1-2{\mathbf{n}_y}^2 & -2\mathbf{n}_y\mathbf{n}_z \\ -2\mathbf{n}_x\mathbf{n}_z & -2\mathbf{n}_y\mathbf{n}_z & 1-2{\mathbf{n}_z}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1-2(.577)^2 & -2(.577)(-.577) & -2(.577)(.577) \\ -2(.577)(-.577) & 1-2(-.577)^2 & -2(-.577)(.577) \\ -2(.577)(.577) & -2(-.577)(.577) & 1-2(.577)^2 \end{bmatrix} \\
&= \begin{bmatrix} .333 & .667 & -.667 \\ .667 & .333 & .667 \\ -.667 & .667 & .333 \end{bmatrix}
\end{aligned}
$$

The vector [-99, 99, -99]?

This is a trick question. The plane perpendicular to this vector is the same plane that is perpendicular to the vector [99, -99, 99], since the two vectors are negatives of each other. Thus, the same matrix can be used to reflect about the plane. If you examine the matrices for scale, projection, and reflection, (Equations 8.9, 8.16, and 8.18, respectively), you will see that in each case, negating **n** results in no change to the matrix.

11) *Does the matrix below express a linear transformation? Affine?*

$$
\begin{bmatrix} 34 & 1.7 & \pi \\ \sqrt{2} & 0 & 18 \\ 4 & -9 & -1.3 \end{bmatrix}
$$

Another trick question. The transformation is both linear and affine. From Section 8.8.1, we know that *any* matrix represents a linear transformation. From Section 8.8.2, we know that any linear transformation is an affine transformations.

Chapter 9

1) *Compute the determinant of the following matrix:*

$$\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$$

We use Equation 9.1 from page 125.

$$\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = (3)(4) - (-2)(1) = 14$$

2) *Compute the determinant, adjoint, and inverse of the following matrix:*

$$\begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The determinant is given by Equation 9.2 from page 126.

$$\begin{aligned} \begin{vmatrix} 3 & -2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} &= \begin{array}{l} (3)\left((4)(2) - (0)(0)\right) \\ \; + (-2)\left((0)(0) - (1)(2)\right) \\ \; + (0)\left((1)(0) - (4)(0)\right) \end{array} \\ &= (3)(8) + (-2)(-2) + (0)(0) \\ &= 28 \end{aligned}$$

To compute the classical adjoint, we first compute the cofactors of **M**. (See Section 9.2.1.)

$$c_{11} = +\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 \quad c_{12} = -\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -2 \quad c_{13} = +\begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} = 0$$

$$c_{21} = -\begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \quad c_{22} = +\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6 \quad c_{23} = -\begin{vmatrix} 3 & -2 \\ 0 & 0 \end{vmatrix} = 0$$

$$c_{31} = +\begin{vmatrix} -2 & 0 \\ 4 & 0 \end{vmatrix} = 0 \quad c_{32} = -\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0 \quad c_{33} = +\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 14$$

Now the adjoint is the transpose of the matrix of cofactors:

$$\text{adj } \mathbf{M} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} 8 & -2 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 14 \end{bmatrix}^T = \begin{bmatrix} 8 & 4 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

To compute the inverse, we divide the classical adjoint by the determinant. (Equation 9.7 on page 131.)

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{\text{adj } \mathbf{M}}{|\mathbf{M}|} \\ &= \frac{\begin{bmatrix} 8 & 4 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 14 \end{bmatrix}}{28} \\ &= \begin{bmatrix} 2/7 & 1/7 & 0 \\ -1/14 & 3/14 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \end{aligned}$$

3) *Is the following matrix orthogonal?*

$$\begin{bmatrix} -0.1495 & -0.1986 & 0.9685 \\ -0.8256 & 0.5640 & 0.0117 \\ -0.5439 & -0.8015 & 0.2484 \end{bmatrix}$$

We can use the ideas from Section 9.3.2 to test the matrix to see if it is orthogonal.

$$\begin{aligned} (-0.1495)(-0.1495) + (-0.1986)(-0.1986) + (0.9685)(0.9685) &= 0.99978446 \\ (-0.1495)(-0.8256) + (-0.1986)(0.5640) + (0.9685)(0.0117) &= 0.02274825 \\ (-0.1495)(-0.5439) + (-0.1986)(-0.8015) + (0.9685)(0.2484) &= 0.48106635 \\ (-0.8256)(-0.1495) + (0.5640)(-0.1986) + (0.0117)(0.9685) &= 0.02274825 \\ (-0.8256)(-0.8256) + (0.5640)(0.5640) + (0.0117)(0.0117) &= 0.99984825 \\ (-0.8256)(-0.5439) + (0.5640)(-0.8015) + (0.0117)(0.2484) &= -0.00009588 \\ (-0.5439)(-0.1495) + (-0.8015)(-0.1986) + (0.2484)(0.9685) &= 0.48106635 \\ (-0.5439)(-0.8256) + (-0.8015)(0.5640) + (0.2484)(0.0117) &= -0.00009588 \\ (-0.5439)(-0.5439) + (-0.8015)(-0.8015) + (0.2484)(0.2484) &= 0.99993202 \end{aligned}$$

The matrix is not orthogonal. The 2nd, 3rd, and 7th sums should be zero, and the 4th sum should be one.

Unfortunately, the matrix was *intended* to be orthogonal! There was a typo in the exercise, element m_{13} was missing a minus sign and should have been –0.9685. With that correction, we compute the nine equations again:

$$\begin{aligned}
(-0.1495)(-0.1495) + (-0.1986)(-0.1986) + (-0.9685)(-0.9685) &= 0.99978446 \\
(-0.1495)(-0.8256) + (-0.1986)(0.5640) + (-0.9685)(0.0117) &= 0.00008535 \\
(-0.1495)(-0.5439) + (-0.1986)(-0.8015) + (-0.9685)(0.2484) &= -0.00008445 \\
(-0.8256)(-0.1495) + (0.5640)(-0.1986) + (0.0117)(-0.9685) &= 0.00008535 \\
(-0.8256)(-0.8256) + (0.5640)(0.5640) + (0.0117)(0.0117) &= 0.99984825 \\
(-0.8256)(-0.5439) + (0.5640)(-0.8015) + (0.0117)(0.2484) &= -0.00009588 \\
(-0.5439)(-0.1495) + (-0.8015)(-0.1986) + (0.2484)(-0.9685) &= -0.00008445 \\
(-0.5439)(-0.8256) + (-0.8015)(0.5640) + (0.2484)(0.0117) &= -0.00009588 \\
(-0.5439)(-0.5439) + (-0.8015)(-0.8015) + (0.2484)(0.2484) &= 0.99993202
\end{aligned}$$

This time, we see that the nine sums are close enough (within tolerance of the precision that we used in the original matrix) to consider the matrix orthogonal.

I apologize for this error.

4) *Invert the matrix from the previous exercise.*

<Cringe> If you took the time to invert the matrix with the typo, then kudos to you! I hope you got this answer:

$$\begin{bmatrix} -0.1495 & -0.1986 & 0.9685 \\ -0.8256 & 0.5640 & 0.0117 \\ -0.5439 & -0.8015 & 0.2484 \end{bmatrix}^{-1} = \begin{bmatrix} .1706 & -.8297 & -.6261 \\ .2268 & .5588 & -.9106 \\ 1.105 & -.0135 & -.2834 \end{bmatrix}$$

The intent of the exercise was not to make you grind through a bunch of math, but for you to realize that since the matrix is orthogonal, the inverse is simply the transpose:

$$\begin{bmatrix} -0.1495 & -0.1986 & -0.9685 \\ -0.8256 & 0.5640 & 0.0117 \\ -0.5439 & -0.8015 & 0.2484 \end{bmatrix}^{-1} = \begin{bmatrix} -0.1495 & -0.8256 & -0.5439 \\ -0.1986 & 0.5640 & -0.8015 \\ -0.9685 & 0.0117 & 0.2484 \end{bmatrix}$$

5) *Invert the 4x4 matrix:*

$$\begin{bmatrix} -0.1495 & -0.1986 & 0.9685 & 0 \\ -0.8256 & 0.5640 & 0.0117 & 0 \\ -0.5439 & -0.8015 & 0.2484 & 0 \\ 1.7928 & -5.3116 & 8.0151 & 1 \end{bmatrix}$$

The typo from the previous problem propogated into this exercise. Luckily, it doesn't really make the problem any more complicated, since the point of this exercise is not really to go through the work of inverting a 4x4 matrix, but to realize that you can use most of your results from the previous exercise.

Recall from Section 9.4.2 (see page 138) that when the righthand column of a 4x4 matrix is $[\ 0, 0, 0, 1\ ]^T$, we can separate the matrix into a 3D linear transform matrix **R**, and a translation matrix **T**. Let **M** be the matrix above (the one from the book, with the typo). Then we have:

$$
\begin{aligned}
\mathbf{R} &= \begin{bmatrix} -0.1495 & -0.1986 & 0.9685 & 0 \\ -0.8256 & 0.5640 & 0.0117 & 0 \\ -0.5439 & -0.8015 & 0.2484 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\mathbf{T} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1.7928 & -5.3116 & 8.0151 & 1 \end{bmatrix} \\
\mathbf{M} &= \begin{bmatrix} -0.1495 & -0.1986 & 0.9685 & 0 \\ -0.8256 & 0.5640 & 0.0117 & 0 \\ -0.5439 & -0.8015 & 0.2484 & 0 \\ 1.7928 & -5.3116 & 8.0151 & 1 \end{bmatrix} \\
&= \mathbf{RT}
\end{aligned}
$$

By breaking **M** into its component parts like this, we have found a shortcut for computing the inverse of **M**. Recall from 9.2.1 (page 131) that the inverse of a matrix product is the product of the inverses taken in reverse order:

$$
\begin{aligned}
\mathbf{M} &= \mathbf{RT} \\
\mathbf{M}^{-1} &= (\mathbf{RT})^{-1} \\
&= \mathbf{T}^{-1}\mathbf{R}^{-1}
\end{aligned}
$$

Since **T** is a translation matrix, $\mathbf{T}^{-1}$ is simply the matrix which translates by the opposite amount. $\mathbf{R}^{-1}$ comes from the previous exercise. The matrix multiplication is easy since there are so many 1's and 0's:

$$
\begin{aligned}
\mathbf{T}^{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1.7928 & 5.3116 & -8.0151 & 1 \end{bmatrix} \\
\mathbf{R}^{-1} &= \begin{bmatrix} .1706 & -.8297 & -.6261 & 0 \\ .2268 & .5588 & -.9106 & 0 \\ 1.105 & -.0135 & -.2834 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\mathbf{M}^{-1} &= \mathbf{T}^{-1}\mathbf{R}^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1.7928 & 5.3116 & -8.0151 & 1 \end{bmatrix} \begin{bmatrix} .1706 & -.8297 & -.6261 & 0 \\ .2268 & .5588 & -.9106 & 0 \\ 1.105 & -.0135 & -.2834 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} .1706 & -.8297 & -.6261 & 0 \\ .2268 & .5588 & -.9106 & 0 \\ 1.105 & -.0135 & -.2834 & 0 \\ -7.9608 & 4.5637 & -1.4431 & 1 \end{bmatrix}
\end{aligned}
$$

6) *Construct a 4x4 matrix to translate by [4,2,3].*

See Equation 9.10 on page 137.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 3 & 1 \end{bmatrix}
$$

7) *Construct a 4x4 matrix to rotate 20° about the* x*-axis and then translate by [4,2,3].*

The upper 3x3 portion of the rotation matrix is constructed using Equation 8.2 on page 108.

$$
\begin{aligned}
\mathbf{R}_x(20^\circ) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 20^\circ & \sin 20^\circ & 0 \\ 0 & -\sin 20^\circ & \cos 20^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .9397 & .3420 & 0 \\ 0 & -.3420 & .9397 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Concatenating this with our matrix from the previous exercise, we get:

$$\begin{aligned}\mathbf{M} &= \mathbf{R}_x(20^\circ)\mathbf{T} \\ &\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .9397 & .3420 & 0 \\ 0 & -.3420 & .9397 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 3 & 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .9397 & .3420 & 0 \\ 0 & -.3420 & .9397 & 0 \\ 4 & 2 & 3 & 1 \end{bmatrix}\end{aligned}$$

8) *Construct a 4x4 matrix to translate by [4,2,3] and then rotate 20° about the* x*-axis.*

We use the same matrices from Exercise 7, only we concatenate them in the opposite order. Notice that only the last row is effected.

$$\begin{aligned}\mathbf{M} &= \mathbf{T}\mathbf{R}_x(20^\circ) \\ &\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .9397 & .3420 & 0 \\ 0 & -.3420 & .9397 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .9397 & .3420 & 0 \\ 0 & -.3420 & .9397 & 0 \\ 4 & .8534 & 3.5031 & 1 \end{bmatrix}\end{aligned}$$

9) *Construct a 4x4 matrix to perform a perspective projection onto the plane* x*=5. (Assume the origin is the center of projection.)*

Equation 9.13, which gives a matrix to project onto the plane $z=d$, was presented in Section 9.4.6. We can apply the same basic principle to project onto a plane of constant x:

$$\begin{bmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

10) *Use the matrix from the previous exercise to compute the 3D coordinates of the projection of the point (107, -243, 89) onto the plane* x*=5.*

First, we extend the point into 4D and compute the projected point in homegenous coordinates.

$$\begin{bmatrix} 107 & -243 & 89 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 107 & -243 & 89 & 107/5 \end{bmatrix}$$

Now we divide by the homegenous coordinate *w* to get the physical 3D coordinates:

$$\begin{aligned} \begin{bmatrix} 107 & -243 & 89 & 107/5 \end{bmatrix} &\Rightarrow \begin{bmatrix} \frac{107}{107/5} & \frac{-243}{107/5} & \frac{89}{107/5} \end{bmatrix} \\ &= \begin{bmatrix} 5 & -\frac{1215}{107} & \frac{445}{107} \end{bmatrix} \\ &\approx \begin{bmatrix} 5 & -11.355 & 4.159 \end{bmatrix} \end{aligned}$$

Chapter 10

1) *Construct a quaternion to rotate 30˚ about the x-axis.*

Using Equation 10.4 on page 162:

$$\begin{aligned} \theta &= 30^o \\ \mathbf{n} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ \mathbf{q} &= \begin{bmatrix} \cos(\theta/2) & \begin{pmatrix} \sin(\theta/2)\mathbf{n}_x & \sin(\theta/2)\mathbf{n}_y & \sin(\theta/2)\mathbf{n}_z \end{pmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \cos(30^o/2) & \begin{pmatrix} \sin(30^o/2)(1) & \sin(30^o/2)(0) & \sin(30^o/2)(0) \end{pmatrix} \end{bmatrix} \\ &\approx \begin{bmatrix} .9659 & \begin{pmatrix} .2588 & 0 & 0 \end{pmatrix} \end{bmatrix} \end{aligned}$$

What is the magnitude of this quaternion?

Since we know the quaternion is a valid rotation quaternion, it is a unit quaternion and therefore the magnitude is one. However, we can verify this using Equation 10.6 from page 163.

$$\begin{aligned} \left\| \begin{bmatrix} .9659 & \begin{pmatrix} .2588 & 0 & 0 \end{pmatrix} \end{bmatrix} \right\| &= \sqrt{.9659^2 + .2588^2 + 0^2 + 0^2} \\ &= \sqrt{.9330 + .06698} \\ &= \sqrt{.99998} \\ &\approx 1 \end{aligned}$$

What is its conjugate?

As per Equation 10.7 on page 164, to obtain the quatnion conjugate, we negate the vector portion:

$$\begin{bmatrix} .9659 & \begin{pmatrix} .2588 & 0 & 0 \end{pmatrix} \end{bmatrix}^* = \begin{bmatrix} .9659 & \begin{pmatrix} -.2588 & 0 & 0 \end{pmatrix} \end{bmatrix}$$

What type of rotation is expressed by the conjugate?

In Section 10.4.7 we learned that the quaternion conjugate represents the opposite rotation as the original quaternion. Since the original quaternion rotates 30˚ about the *x*-axis, the conjugate rotates *negative* 30˚ about the *x*-axis. In the next exercise, we show how to extract the angle and axis of rotation manually.

2) *What type of rotation is represented by the quaternion:*

$$\begin{bmatrix} 0.965 & \begin{pmatrix} 0.149 & -0.149 & 0.149 \end{pmatrix} \end{bmatrix}$$

We apply Equation 10.4 from page 162 in reverse. First, we extract the angle of rotation, θ, from the w component of the quaternion:

$$\begin{aligned} w &= \cos(\theta/2) \\ .965 &= \cos(\theta/2) \\ \text{acos}\ (.965) &= \theta/2 \\ 15.20^\circ &= \theta/2 \\ 30.40^\circ &= \theta \end{aligned}$$

Now that we have the angle of rotation, we can solve for the axis of rotation, **n**:

$$\begin{aligned} \mathbf{v} &= \sin(\theta/2)\mathbf{n} \\ \begin{bmatrix} .149 & -.149 & .149 \end{bmatrix} &= \sin(30.40^\circ/2)\mathbf{n} \\ \begin{bmatrix} .149 & -.149 & .149 \end{bmatrix} &= .2622\mathbf{n} \\ \begin{bmatrix} .149 & -.149 & .149 \end{bmatrix}/.2622 &= \mathbf{n} \\ \begin{bmatrix} .568 & -.568 & .568 \end{bmatrix} &= \mathbf{n} \end{aligned}$$

The actual result is closer to [.57735, -.57735, .57735]. Because we were using only three decimal digits, the floating point error has accumulated. (In fact, the angle of rotation was actually 30 degrees, but the limited precision caused large roundoff errors, due to the inverse trig functions, which are highly non-linear.)

Compute a quaternion which performs 1/5th of this rotation.

In Section 10.4.12, we learned about quaternion exponentiation, which is used to compute a quaternion which represents a "fraction" of the rotation of a given quaternion. Officially, this uses the quaternion exp and log operations, as shown in Equation 10.18. However, as we mentioned in that same section, the quaternion log and exp operations are nice mathematical formalities, but in practice, the quaternion expoentiation is computed by extracting the rotation angle and axis, taking the desired fraction of the rotation angle, and then computing a new quaternion. This technique is illustrated in code in Listing 10.1. Now we'll apply it mathematically in an example. (We'll use the more precise values.)

$$\begin{aligned}
\theta &= 30^{\circ} \\
\mathbf{n} &= \begin{bmatrix} .57735 & -.57735 & .57735 \end{bmatrix} \\
\theta' &= (1/5)\theta \\
&= 6^{\circ} \\
\mathbf{q}' &= \begin{bmatrix} \cos(\theta'/2) & \sin(\theta'/2)\mathbf{n} \end{bmatrix} \\
&\approx \begin{bmatrix} .99452 & .10453\mathbf{n} \end{bmatrix} \\
&\approx \begin{bmatrix} .99452 & \begin{pmatrix} .06035 & -.06035 & .06035 \end{pmatrix} \end{bmatrix}
\end{aligned}$$

3) *Consider the quaternions.*

$$\begin{aligned}
\mathbf{a} &= \begin{bmatrix} 0.233 & \begin{pmatrix} 0.060 & -0.257 & -0.935 \end{pmatrix} \end{bmatrix} \\
\mathbf{b} &= \begin{bmatrix} -0.752 & \begin{pmatrix} 0.286 & 0.374 & 0.459 \end{pmatrix} \end{bmatrix}
\end{aligned}$$

Compute the dot product ***a·b****.*

Using Equation 10.14 on page 169:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \begin{bmatrix} .233 \\ \begin{pmatrix} .060 \\ -.257 \\ -.935 \end{pmatrix} \end{bmatrix} \cdot \begin{bmatrix} -.752 \\ \begin{pmatrix} .286 \\ .374 \\ .459 \end{pmatrix} \end{bmatrix} \\
&= (.233)(-.752) + (.060)(.286) + (-.257)(.374) + (-.935)(.459) \\
&= -.1752 + .01716 + -.09612 + -.4292 \\
&= -.6834
\end{aligned}$$

Compute the difference from ***a*** *to* ***b***

See Section 10.4.9

$$
\begin{aligned}
\mathbf{a}^{-1}\mathbf{b} &= \begin{bmatrix} .233 \\ \begin{pmatrix} .060 \\ -.257 \\ -.935 \end{pmatrix} \end{bmatrix}^{-1} \begin{bmatrix} -.752 \\ \begin{pmatrix} .286 \\ .374 \\ .459 \end{pmatrix} \end{bmatrix} \\
&= \begin{bmatrix} .233 \\ \begin{pmatrix} -.060 \\ .257 \\ .935 \end{pmatrix} \end{bmatrix} \begin{bmatrix} -.752 \\ \begin{pmatrix} .286 \\ .374 \\ .459 \end{pmatrix} \end{bmatrix} \\
&= \begin{bmatrix} (.233)(-.752) - (-.060)(.286) - (.257)(.374) - (.935)(.459) \\ \begin{pmatrix} (.233)(.286) + (-.060)(-.752) + (.935)(.374) - (.257)(.459) \\ (.233)(.374) + (.257)(-.752) + (-.060)(.459) - (.935)(.286) \\ (.233)(.459) + (.935)(-.752) + (.257)(.286) - (-.060)(.374) \end{pmatrix} \end{bmatrix} \\
&\approx \begin{bmatrix} -.6833 \\ \begin{pmatrix} .3435 \\ -.4011 \\ -.5002 \end{pmatrix} \end{bmatrix}
\end{aligned}
$$

*Compute the quaternion product **ab**.*

Using Equation 10.13 on page 168.

$$
\begin{aligned}
\mathbf{ab} &= \begin{bmatrix} .233 \\ \begin{pmatrix} .060 \\ -.257 \\ -.935 \end{pmatrix} \end{bmatrix} \begin{bmatrix} -.752 \\ \begin{pmatrix} .286 \\ .374 \\ .459 \end{pmatrix} \end{bmatrix} \\
&= \begin{bmatrix} (.233)(-.752) - (.060)(.286) - (-.257)(.374) - (-.935)(.459) \\ \begin{pmatrix} (.233)(.286) + (.060)(-.752) + (-.935)(.374) - (-.257)(.459) \\ (.233)(.374) + (-.257)(-.752) + (.060)(.459) - (-.935)(.286) \\ (.233)(.459) + (-.935)(-.752) + (-.257)(.286) - (.060)(.374) \end{pmatrix} \end{bmatrix} \\
&\approx \begin{bmatrix} .3329 \\ \begin{pmatrix} -.2102 \\ .5754 \\ .7141 \end{pmatrix} \end{bmatrix}
\end{aligned}
$$

4) Convert the quaternion in exercise 2 to matrix form.

Here's the quaternion from exercise 2:

$$\begin{bmatrix} 0.965 & \begin{pmatrix} 0.149 & -0.149 & 0.149 \end{pmatrix} \end{bmatrix}$$

We can convert this quaternion to a matrix in two different ways. The most straightforward way is to apply Equation 10.23 (page 187) directly:

$$
\begin{aligned}
\mathbf{M} &= \begin{bmatrix} 1-2y^2-2z^2 & 2xy+2wz & 2xz-2wy \\ 2xy-2wz & 1-2x^2-2z^2 & 2yz+2wx \\ 2xz+2wy & 2yz-2wx & 1-2x^2-2y^2 \end{bmatrix} \\
&= \begin{bmatrix} 1-2(-.149)^2-2(.149)^2 & 2(.149)(-.149)+2(.965)(.149) & 2(.149)(.149)-2(.965)(-.149) \\ 2(.149)(-.149)-2(.965)(.149) & 1-2(.149)^2-2(.149)^2 & 2(-.149)(.149)+2(.965)(.149) \\ 2(.149)(.149)+2(.965)(-.149) & 2(-.149)(.149)-2(.965)(.149) & 1-2(.149)^2-2(-.149)^2 \end{bmatrix} \\
&\approx \begin{bmatrix} .9112 & .2432 & .3320 \\ -.3320 & .9112 & .2432 \\ -.2432 & -.3320 & .9112 \end{bmatrix}
\end{aligned}
$$

5) *Write the C++ code to convert an object-to-inertial matrix to Euler angle form.*

Listing 10.3 (page 184) presented code to extract Euler angles from an inertial-to-object rotation matrix. From Section 9.3, we know that the object-to-inertial matrix is the transpose of the inertial-to-object matrix. Thus, we can start with listing 10.3, and whenever a matrix element is referenced, replace that matrix element by the corresponding element in the transpose. This results in the following code snippet:

```
// Assume the matrix is stored in these variables:

float  m11,m12,m13;
float  m21,m22,m23;
float  m31,m32,m33;

// We will compute the Euler angle values in radians and store them here:

float  h,p,b;

// Extract pitch from m32, being careful for domain errors with asin().  We could have
// values slightly out of range due to floating point arithmetic.

float sp = -m32;
if (sp <= -1.0f) {
       p = -1.570796f; // -pi/2
} else if (sp >= 1.0) {
       p = 1.570796; // pi/2
} else {
       p = asin(sp);
}

// Check for the Gimbal lock case, giving a slight tolerance
// for numerical imprecision

if (sp > 0.9999f) {

       // We are looking straight up or down.
       // Slam bank to zero and just set heading

       b = 0.0f;
       h = atan2(-m13, m11);

} else {
```

```
        // Compute heading from m31 and m33

        h = atan2(m31, m33);

        // Compute bank from m12 and m22

        b = atan2(m12, m22);
}
```